Acronyme/*short title*	**INTOCS**
Titre du projet *(en français)*	INTeraction d'Ondes CompressibleS
Titre du projet/*Proposal title* *(en anglais)*	Interaction of compressible waves

Les pages seront numérotées et l'acronyme du projet devra figurer sur toutes les pages du document en pied de page.
Un sommaire du document est bienvenu

1 Programme scientifique et technique/Description du projet *Technical and scientific description of the proposal*

1.1 Problème posé/*Rationale* *(1/2 page maximum)*

Présentation générale du problème qu'il est proposé de traiter dans le projet et du cadre de travail dans lequel il sera effectué.

The aim of the project is to study the interaction of waves in compressible inviscid fluid mechanics. Up to now, the mathematical theory for the multidimensional compressible Euler equations deals mainly with solutions that represent a single wave: either a shock wave, a rarefaction wave, or a contact discontinuity. First of all, we wish to study the interaction between such isolated waves and small amplitude high frequency oscillations. Our main goal here is to give a rigorous justification for the formation of singularities such as « kink modes » or « Mach stems » whose existence has been formally explained in the eighties. Such singularities can be responsible for the breakdown of single wave patterns so a rigorous mathematical theory for such blow-up phenomena would give important qualitative information on the solutions to the Euler equations. In a second step, we wish to study the interaction between large amplitude waves. In particular we wish to develop a general stability theory for the interaction of several waves of different families. Our goal is to classify the stable and unstable wave patterns for the Euler equations in two or three space dimensions. Eventually we wish to develop efficient numerical schemes that are able to capture highly oscillating solutions for hyperbolic problems in the whole space or in bounded domains. For boundary value problems, the scheme should be also able to capture small instability phenomena that arise for instance in elastodynamics (Rayleigh waves) or in compressible fluid mechanics (« kink modes » or « Mach stems »). The stability properties of our numerical schemes will be clearly investigated.

1.2 Contexte et enjeux du projet/*Background, objective, issues and hypothesis* *(1 à 3 pages maximum)*

Décrire ***le contexte*** *en dressant un* ***état de l'art national et international*** *incluant les références nécessaires et préciser les* ***enjeux scientifiques*** *du projet.*

The Cauchy problem for the multidimensional compressible Euler equations is one of the major challenges in the theory of hyperbolic systems of conservation laws. In one space dimension, the theory of global weak solutions is rather well understood, with solutions that can be constructed either by the Glimm scheme or by the vanishing viscosity method. In several space dimensions, the theory is far less understood. In particular, stability of particular wave patterns becomes a very intricate physical and mathematical challenge that requires powerful analytical methods. This project aims at studying, both from an analytical and a numerical point of view, new stability issues for the interaction of nonlinear waves for the Euler equations. We recall here some known results on multidimensional systems of conservation laws, and the justification of geometric optics expansions.

Up to now, only a few general results have been obtained on the existence of solutions to multidimensional systems of conservation laws. The first main result is the local existence of smooth solutions that has been obtained by Kato (Archive for Rational Mechanics and Analysis, 1975). This first result can be thought of as a nonlinear stability result for constant solutions. Then there has been some effort to obtain nonlinear stability results for particular « one-dimensional » solutions (these solutions depend on a single space variable but are also solutions to the multidimensional equations). In particular Majda (Memoir of the American Mathematical Society, 1983) obtained stability results for shock waves, Alinhac (Communications on Partial Differential Equations, 1989) obtained stability results for rarefaction waves, Métivier (Journal de Mathématiques Pures et Appliquées, 1991) obtained stability results for sonic waves, and Coulombel and Secchi (Annales Scientifiques de l'Ecole Normale Supérieure, 2005) obtained stability results for contact discontinuities. The list is of course not limitative. As far as solutions with interacting waves are concerned, Métivier (Transactions of the American Mathematical Society, 1986) studied the interaction of two shock waves. Generally speaking, at this stage, the situations that are understood represent hyperbolic boundary value problems for which the boundary conditions lead to maximal energy estimates (the so-called uniform Kreiss condition) or some cases where the boundary conditions lead to energy estimates with a loss of regularity (the so-called weak Kreiss condition). As far as interactions are concerned, there has still not been any study of interactions between waves that do not satisfy maximal energy estimates. The project aims at studying more deeply the stability or instability properties of waves that do not satisfy maximal energy estimates. This is a new field of research since the first general nonlinear stability results for such solutions were only obtained in the past few years. The project focuses on two main problems:

1. The interaction of a weakly stable wave with small amplitude highly oscillating waves.
2. The interaction between a strongly stable wave and a weakly stable wave.

For both problems, the main tool is geometric optics and the so-called WKB expansions. The introduction of such methods in the study of hyperbolic partial differential equations was mainly initiated by Lax (Duke Mathematical Journal, 1957). There has been an impressive quantity of works for highly oscillating solutions to the Cauchy problem (in the whole space). Schematically, the analysis proceeds in two steps. In the first step, one formally constructs an asymptotic solution that takes the form of a power series in the small wavelength parameter. The second step is to justify that the true solution to the Cauchy problem with highly oscillating data is close to the formal asymptotic solution. This general program was achieved by Lax in the case of linear problems. It was then extended to semilinear problems by Joly and Rauch (Transactions of the American Mathematical Society, 1992), and to quasilinear problems by Guès (Asymptotic Analysis, 1993). These works give a rigorous justification for highly oscillating solutions to nonlinear problems before the formation of singularities (shock waves or other types of discontinuity in the solutions). In the case of boundary value problems, the justification of geometric optics expansions has been achieved by Williams in a series of papers (Communications on Partial Differential Equations 1996, Communications on Pure and Applied Mathematics 1999, Journal of Functional Analysis 2002). All the results obtained by Williams deal with situations in which the so-called uniform Kreiss-Lopatinskii condition is satisfied. In this case, one can prove maximal energy estimates that justify the relevance of WKB approximations for highly oscillating solutions. When the uniform Kreiss-Lopatinskii condition is not satisfied, hyperbolic boundary value problems lead to some weak instability phenomena. At the formal level, Majda and Rosales (SIAM Journal on Applied Mathematics, 1983) then Artola and

Majda (Physica D, 1987) constructed the first terms of a WKB expansion for highly oscillating solutions close to a weakly stable wave (either a shock wave or a contact discontinuity). The main phenomenon that arises in this context is the amplification of small amplitude highly oscillating waves when they are reflected by a discontinuity in the solution. This amplification phenomenon is responsible for the formation of singularities that form in finite time. In the context of reacting shock waves these singularities are called « Mach stems », while for compressible vortex sheets the singularities are « kink modes ». These singularities in the solutions then lead to several new interaction phenomena between the main underlying discontinuity and the new small discontinuities that emanate from the singularity (once it has appeared). At the theoretical level, the main goal of the project is to give a complete rigorous justification for geometric optics expansions in the context of boundary value problems that do not satisfy the uniform Kreiss-Lopatinskii condition. For single wave solutions to the Euler equations, such results will justify the formation of singularities described above. For multiple waves solutions to the Euler equations, these first results will be a basis for a general stability theory for interacting waves patterns.

For linear problems, the stability of finite difference schemes (or finite volume schemes on a uniform grid) can be efficiently studied by means of Fourier analysis. For boundary value problems, there has been a series of works initiated by Kreiss (Mathematics of Computation, 1968) that study the stability of finite difference approximations for boundary value problems. The original work by Kreiss was later refined by Gustafsson, Kreiss and Sundstrom (Mathematics of Computation, 1972) then by Goldberg and Tadmor (Mathematics of Computation, 1978, 1981, 1985 and 1987). The theory was extended to multidimensional problems by Michelson (Mathematics of Computation, 1983). However, all these works are often restricted by strong assumptions on the finite difference approximation of the hyperbolic operator. From a practical point of view, the goal of numerical simulations of hyperbolic problems is mainly to get the lowest possible numerical diffusion, so it would be convenient to obtain a nice general stability theory for finite difference approximations of boundary value problems with the least possible dissipation assumptions. If one wants to simulate highly oscillating solutions, it is crucial to obtain the lowest diffusive numerical schemes in order to compute accurate amplitudes and phases for the solutions. The project will focus on the construction and analysis of numerical schemes for highly oscillating solutions to hyperbolic boundary value problems. A correct analysis of the frequency limitations of the computation grid as well as the correct reflection of oscillations on the boundary are the main goals to achieve, first for linear problems then for nonlinear problems. This seems to be a new field for research in scientific computing and numerical analysis, that is motivated by numerous examples in continuum mechanics (elastodynamics, fluid mechanics) for which the boundary conditions lead to weakly stable phenomena. After clarifying the theory and numerical simulations for linear problems and finite difference approximations, the main goal of the project on the numerical side is to obtain accurate simulations for small amplitude highly oscillating solutions to the Euler equations. The goal of the simulations is to reproduce the « kink modes » formation predicted by the theoretical analysis, and to obtain quantitative estimates for the time at which these singularities form. Elastodynamics (with the Rayleigh waves) and reacting shock fronts (with the « Mach stems ») provide with other test cases that will be studied in this context.

1.3 Objectifs et caractère ambitieux/novateur du projet/*Specific aims, highlight of the originality and novelty of the project* *(1 à 2 pages maximum)*

*Décrire les **objectifs scientifiques/technologiques** du projet.*
*Présenter l'**avancée scientifique** attendue. Préciser l'originalité et les ambitions du projet.*
*Détailler les **verrous scientifiques et technologiques** à lever par la réalisation du projet.*

As quickly described above, the main scientific goal of the project is to study, both on the theoretical and numerical point of view, geometric optics expansions for hyperbolic boundary value problems.

The main application we have in mind is compressible fluid mechanics. Achieving our goals will clearly give new insight on various important problems that appear in the theory of hyperbolic systems of conservation laws. We detail them below.

First of all, a rigorous justification for the formation of singularities like « kink modes » or « Mach stems » will be one the first qualitative descriptions of a blow-up phenomenon for solutions to multidimensional systems . Up to now, blow-up criteria for smooth solutions or shock waves are quantitative descriptions that inform us on the divergence of some norms of the solutions, but they do not really give « pointwise information » on the divergence of the solutions. Justifying the pointwise formation of singularities is a new approach that could be extended to other situations. As we shall explain more precisely below, the construction and justification of geometric optics expansions in this context may also be a new mathematical challenge because of the amplification of waves at the boundary. This seems to be a nonstandard phenomenon in WKB expansions that will probably require sophisticated analytical tools for a rigorous justification.

As far as the interaction of large waves is concerned, our current knowledge of multiple waves patterns is still very poor. Following Métivier's result for the interaction of two uniformly stable shock waves, several works have dealt with the interaction of uniformly stable shock waves and rarefaction waves. However, there has not been any study of the interaction between a uniformly stable discontinuity and a weakly stable discontinuity (for instance a vortex sheet in compressible fluid mechanics). Our goal is to face this new challenge that is now open for investigation thanks to our better knowledge of the stability properties for contact discontinuities. A similar question arises for boundary value problems in a quarter-space or a more general region with a corner. The clarification of such problems will undoubtedly give new stability or instability results for various interaction patterns. Understanding new interaction phenomena is also a natural step towards trying to solve the global-in-time Cauchy problem for the Euler equations.

On the numerical side, the effective computation of highly oscillating solutions has received much attention in various contexts such as laser beams. The originality of our problem is to couple the numerical difficulties of highly oscillating wave propagation with a sharp treatment of the boundary conditions. Another specificity of our problems is their weakly stable nature that may cause an amplification for the incoming waves that are reflected at the boundary. We wish as much as possible to check that our numerical results are in good agreement with the main features predicted by the theory. The numerical validation of the formation of singularities or the development of instabilities may also give some insight to tackle future analytical problems.

An important specificity of the project is to face both theoretical and computational aspects at the same time in strong connection one with the other.

1.4 Description des travaux : programme scientifique/*For each specific aim: a proposed work plan should be described (including preliminary data, work packages and deliverables)* *(10 pages maximum)*

Décrire le ***programme de travail décomposé par tâches en cohérence avec les objectifs poursuivis****. Les tâches représentent les grandes phases du projet. Elles sont en nombre restreint.*

Pour chaque tâche, préciser :

- *les objectifs de la tâche*
- *le programme détaillé des travaux correspondants.*

As should be clear from the preceeding paragraphs, the project aims at studying three main problems. These problems are detailed below with their own specific program.

I) Interaction of a weakly stable wave with small amplitude highly oscillating waves

This first subject is motivated by some previous formal works by Artola and Majda (Physica D, 1987) who have computed WKB expansions for small amplitude highly oscillating solutions of the vortex sheets problem for the two-dimensional compressible Euler equations. Vortex sheets can be at most weakly stable so at the time of Artola and Majda's study there was no local in time existence theory for compressible vortex sheets. Such an existence theory is available now. Artola and Majda's analysis indicates that the main term of the WKB expansion for the vortex sheet location satisfies a Burgers type equation that is responsible for the formation of the so-called « kink modes ». These kink modes are singularities in the vortex sheet. Moreover, an important phenomenon that is highlighted in Artola and Majda's work is that high frequency waves of frequency $1/\varepsilon$ and amplitude ε^2 can be reflected by the vortex sheet into waves with the same frequency but with amplitude ε. This amplification is more or less equivalent to the loss of regularity that appears in the energy estimates for the solvability of the vortex sheets problem. Once kink modes are formed, they give rise to new interactions between the main underlying vortex sheet and small shocks or rarefactions. Similar asymptotic expansions have been obtained in the case of reacting shock waves by Majda and Rosales (SIAM Journal on Applied Mathematics, 1983) to explain the « Mach stems » phenomenon, so a complete study of such asymptotic expansions may help to understand the formation of singularities in various situations of compressible fluid dynamics. The first main analytical task of the project is therefore to give a complete and rigorous justification of such highly oscillatory solutions for weakly stable hyperbolic boundary value problems. A complete justification of geometric optics expansions for hyperbolic boundary value problems has been given by Williams (Communications in Partial Differential Equations, 1996) in the case where the uniform Kreiss condition is satisfied. Here we want to generalize these results to the weak Kreiss condition case.

The first model problem will consist in a linear constant coefficients problem. Such a model case will indicate how the WKB expansion is constructed, and how such an asymptotic expansion may be rigorously justified. The computations by Artola and Majda show that small amplitude waves can be amplified while reflected by the boundary. This indicates a strong connection between the various scales in the WKB expansion. Such connections do not appear very clearly in Artola and Majda's work because only the first term of the WKB expansion is constructed. We therefore need to clarify the construction of a complete WKB expansion, and the model of a constant coefficients problem is clearly the first problem to deal with. The connection between various scales of the WKB expansion is clearly a new technical obstacle that we shall need to overcome, because in general WKB expansions are constructed by solving an induction relation that proceeds like a « lower traingular system » (one constructs the first term that satisfies a single equation, then the first term helps constructing the second one and so on). Here we shall need to understand what is the appropriate ansatz for the WKB expansion, and how this ansatz can be rigorously constructed. The next step is to justify the validity of the WKB expansion, and this step uses stability estimates that are already available in the litterature. Let us observe that a complete justification of WKB expansions may also indicate the optimality of energy estimates for weakly stable hyperbolic boundary value problems. In the uniform Kreiss condition case, the maximal energy estimates are clearly optimal, but the situation is not so clear in the weak Kreiss condition case: is the loss of regularity concentrated on the boundary of the domain or is there also an unavoidable loss of regularity in the interior domain ?

Once the constant coefficients situation is clarified, we shall be able to face linear variable coefficients and semilinear situations. The third and final step consists in quasilinear situations, such as the vortex sheets problem (Artola and Majda) or the case of reacting shock fronts (Majda and Rosales). Some members of the project are specialists in geometric optics and some have obtained results on hyperbolic boundary value problems. The combination of both skills is a key point to achieve the expected results in this section. If one keeps in mind the derivation of energy estimates for hyperbolic boundary value problems, the major difficulty is the symbolic analysis of constant coefficients problems. If we are able to cope with the first main step in this section, then the extension to variable coefficients and nonlinear situations is mainly a matter of quantification and using the appropriate functional framework. At this stage, we may hope that some results in the litterature may be very useful to extend the constant coefficients analysis to more realistic situations.

II) Interaction between a strongly stable wave and a weakly stable wave

This second main subject of the project is the natural following step in the study of the Cauchy problem for the Euler equations, and more generally for hyperbolic systems of conservation laws. Keeping in mind that the general Riemann problem gives rise to the interaction of a shock, a contact discontinuity and a rarefaction, we wish to study the stability of some interactions between two waves (the general case of three or more interacting waves seems out of reach right now, and « binary » interactions already constitute a strong mathematical challenge). The main task in this subject is to understand the interaction between a shock (either a strongly stable one or a weakly stable one) and a contact discontinuity. More generally speaking, we want to study the stability or instability properties of solutions that consist in two interacting discontinuities, one of which satisfying only the weak Kreiss condition. In terms of geometric optics, we know that small amplitude high frequency waves are reflected and possibly amplified by a weakly stable discontinuity. In the case of a corner between two waves, we need to understand whether rays may accumulate at the corner and be amplified more than once. Multiple reflections and amplifications may be the cause for a strong instability of the wave pattern. Highlighting such a phenomenon would constitute a completely new type of result for the compressible Euler equations (up to now, results focus more on stability properties of solutions rather than on instability properties). Of course, this second main subject can be attacked only after a complete study of geometric optics expansions for a single weakly stable wave solution.

The method that we propose to study interacting waves mimics the method proposed in the preceeding paragraph. We wish to justify energy estimates and geometric optics expansions for hyperbolic boundary value problems in regions with corners. The simplest case is a constant coefficients hyperbolic boundary value problem in a quarter space (here the space dimension equals 2). Even in the case where the boundary conditions on each side of the corner satisfy the uniform Kreiss condition, Osher (Transactions of the American Mathematical Society, 1973 and 1974) has shown that the overall boundary value problem does not necessarily meet a maximal energy estimate. In two space dimensions, our first task is to derive sufficient structural conditions that imply energy estimates in the case where the boundary conditions on one face of the corner only satisfies the weak Kreiss condition. Deriving necessary and sufficient conditions is certainly much more significant, but considerably harder. Our goal is to derive sufficient conditions that can be applied to a large class of problems. If we can develop such a method for boundary value problems in a fixed domain with corners, then the next step in our analysis would be to extend the theory to interacting waves. The main application we have in mind is the shock-vortex sheet interaction already mentioned earlier. It is still very hard to predict the type of result (stability or instability) that can be obtained for such a problem, even in the case of linear constant coefficients problems. Only a careful analysis of the simplest situation may give a hint of what happens for interactions involving weakly stable waves. The goal of this part of the project is to clarify this new class of hyperbolic boundary value problems that may reveal new aspects in the theory of compressible fluid dynamics.

III) Numerical computations of boundary value problems, high frequency waves and interacting waves

There exists a wide choice of numerical methods to simulate the evolution of a compressible fluid according to the Euler equations. On regular grids, multidimensional finite volume methods are often based on a dimensional splitting. In the case of boundary value problems, most schemes require a suitable discretization of the boundary conditions together with extrapolation techniques for the outgoing components of the solution. The first main task of this section is to construct efficient numerical schemes for multidimensional boundary value problems, whose stability properties reproduce the stability properties of the continuous equation. More precisely, boundary conditions that satisfy the uniform Kreiss condition should give a uniformly stable numerical scheme (that satisfies the discrete analogue of the continuous maximal estimates), and boundary conditions that satisfy the weak Kreiss condition should give a weakly stable numerical scheme. Up to now, the stability analysis of numerical schemes has been performed for dissipative schemes, while the usual goal of numerical simulations is mainly to achieve the lowest numerical dissipation in order to capture sharp singularities. Here we want to construct a low-dissipative scheme with good stability properties. The construction of such a « boundary conditions independent » numerical scheme is an important step towards the numerical simulation of high frequency waves.

Indeed, the following task in this subject is to simulate high frequency waves for hyperbolic boundary value problems. The goal of such numerical simulations is to observe the phenomena that are predicted by the theoretical results. For uniformly stable boundary conditions, we expect the numerical scheme to reproduce the reflection (with no amplification) and propagation of high frequency waves at the boundary, while for weakly stable boundary conditions we expect the scheme to achieve both reflection and amplification of the waves. The main task for this subject is both to construct the numerical scheme and to understand the range of numerical parameters for which the theoretical WKB expansions can be numerically reproduced. In particular, for a fixed mesh size, what are the maximal amplitude and frequency of solutions that can be handled by the scheme ? Once the model situation is clarified, the next task is to derive numerical schemes that are able to capture the expected singularities for the compressible Euler equations: « kink modes », or even « Mach stems » in the reacting case. This is a considerable step forward because the simulation of nonlinear phenomena requires an increased computational effort and in our problem we wish to compute solutions with a very fine structure. The purpose of the numerical tests is reproduce and confirm the theoretical predictions given by the WKB expansion.

In connection with the second main theoretical subject of the project, the last numerical task is to obtain efficient simulations for hyperbolic boundary value problems in regions with corners. There are still very few theoretical foundations for such problems so it is hard to predict what types of results are to be expected. Accurate numerical simulations will clearly give a possible hint of the phenomena that are involved. There are several known explicit examples of ill-posed boundary value problems so we already have some good numerical tests to begin with. If the simulation of high frequency waves in half-spaces can be performed with good numerical results, then we expect to develop some similar methods in regions with corners. Repeated numerical simulations may then detect some possible instability phenomena arising from the amplification of waves near the corner of the space domain.

1.5 Résultats escomptés et retombées attendues/*Expected results and potential impact* (1/2 pages maximum)

*Présenter les **résultats escomptés** en proposant si possible des critères de réussite et d'évaluation adaptés au type de projet, permettant d'évaluer les résultats, tâche par tâche et globalement en fin de projet.*

*Présenter les **retombées attendues** en précisant si possible :*

- *la valorisation des résultats attendus, connaissances à protéger ou à diffuser, …*
- *les retombées scientifiques, techniques, industrielles, économiques…*

Présenter, si besoin, la stratégie de valorisation et de protection des résultats

We summarize here the main goals of the project:

- To obtain a rigorous construction and accurate stability estimates for highly oscillating solutions to linear hyperbolic boundary value problems in the weakly stable case.
- To extend these preliminary linear results to nonlinear problems such as vortex sheets for the Euler equations.
- To study boundary value problems in a quarter-space when the boundary conditions on one edge of the corner are only weakly weel-posed.
- To study the stability of interacting wave patterns in compressible fluid mechanics.
- For all these problems, to construct accurate numerical schemes that reproduce the main qualitative properties of the solutions. For such numerical schemes, to obtain quantitative stability estimates that confirm the numerical simulations.

We expect to make significant progress on both quantitative and qualitative aspects in the theory of multidimensional hyperbolic problems. We also plan to have the widest diffusion of our results in the scientific community through communications in international conferences and giving free access to preprints and numerical tests.